String Theory: A Model Beyond Popular Physics

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Introduction

For those interested in string theory, there is a vast amount of available literature quite accessible to the layperson. However, when one chooses to seek a level of understanding beyond popular physics, the prerequisite knowledge and mathematical sophistication of explanation often renders the subject virtually inaccessible. It is my intention to provide a more involved understanding of the basic ideas and mathematical constructs of bosonic string theory at a level that does not require a mastery of general relativity and quantum field theory.

Beginning with the classical action for a point particle, we will follow a series of logical steps that will, ultimately, illustrate how strings can manifest as a variety of bosons. While this model of string theory lacks fermions and can, therefore, not be the correct model, its purpose as a pedagogical tool can not be underestimated.

Classical Dynamics

We begin with the action of a non-relativistic point particle,

$$S[x]=\int_{t_i}^{t_f}\left\{\frac{1}{2}m(\dot{x}(t))^2-V(x(t))\right\}dt,$$

which is defined as the integral of the Lagrangian. Upon minimizing this functional, we can attain the equation of motion for the point particle, which is easily recognizable as Newton's second law.

Next, we seek the action of a non-relativistic string. This action is more difficult, as the string Lagrangian, itself, involves an integral. For a string stretched over a length *a*, the action is found through some effort to be

$$S=\int_{t_i}^{t_f}dt\int_0^a dx\left[\frac{1}{2}\mu_0\left(\frac{\partial y}{\partial t}\right)^2-\frac{1}{2}T_0\left(\frac{\partial y}{\partial x}\right)^2\right].$$

While minimizing this functional requires that we have additional information, namely the boundary conditions, generalizing this action and requiring that the variation vanish yields

$$\frac{\partial^2 y}{\partial x^2}-\frac{\mu_0}{T_0}\frac{\partial^2 y}{\partial t^2}=0,$$

which is the familiar wave equation.

----- An aside on boundary conditions -----

Two important types of boundary conditions are Dirichlet and Neumann conditions. The latter pertains to the derivative of $y(t,x)$ at the endpoints, and the former refers to the values of $y(t,x)$ at the endpoints. The Dirichlet conditions are especially important in string theory, and they will give rise to the D-branes (Dirichlet branes) that will serve to anchor the ends of the strings!

Relativistic Dynamics

As we continue on to relativistic dynamics, the situation becomes much more complicated, as we must now modify the action in order to account for both the speed limit of light and Lorentz invariance. Starting with

$$S=-mc\int_P ds,$$

and expressing *ds* in terms of *dt* via the Minkowski metric, we find the relativistic action for the point particle to be

$$S=-mc\int_{\tau_i}^{\tau_f}\sqrt{-\eta_{\mu\nu}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial\tau}}\,d\tau,$$

where the action has been parameterized using τ, proper time, as the parameter.

It is of the utmost importance to notice that this action is *reparameterization invariant*. That is, we can easily change our chosen parameter from τ to, say, τ′, without altering the form of the action. This is the mathematical manifestation (a more general one) of Lorentz invariance.

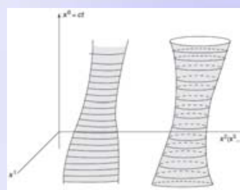

Figure 1 (ref. Zwiebach): Extended objects trace out surfaces as they move through space-time. The figure on the left illustrates two surfaces: one traced out by an open string, the other by a closed string. Open and closed strings will manifest very different particles with very different attributes. It is worth noting that the closed strings, as opposed to open ones, have no well-defined endpoints.

It is in the analysis of an extended relativistic object, e.g. a string, that the situation really begins to get interesting. While a point particle traces out a world-line in space-time, a string traces out a two dimensional world-sheet (Figure 1). In the same way that we parameterized the world-line with respect to a time parameter, we must parameterize the world-sheet with respect to both a time and a space parameter. We will use τ and σ, respectively, for these parameters.

Parameterization can be thought of as a mapping from one surface to another. So, we are, in essence mapping our world-sheet, which is a relatively complicated surface, onto a much simpler surface defined by our chosen parameters… though it is actually the other way around.

Mapping any point in space-time to our new surface,

$$x^\mu=(x^0,x^1,\ldots,x^d)\Rightarrow X^\mu(\tau,\sigma),$$

we can write the Nambu-Goto string action as

$$S=-\frac{T_0}{c}\int_{\tau_i}^{\tau_f}d\tau\int_0^{\sigma_1}d\sigma\sqrt{\left(\frac{\partial X}{\partial\tau}\cdot\frac{\partial X}{\partial\sigma}\right)^2-\left(\frac{\partial X}{\partial\tau}\right)^2\left(\frac{\partial X}{\partial\sigma}\right)^2},$$

where T_0 is the string tension, and $\sigma_1\in\Re>0.$

This equation is non-trivial, but it will yield explicit solutions quite easily if we make good choices for our τ and σ parameterizations.

----- Motivation for the string and T_0 -----

One motivation for considering an extended object, rather than a point, as fundamental stems from the uncertainty principle, i.e. $\Delta x\cdot\Delta p\sim\hbar$. Following this notion, two particles interacting at a point imply infinite momentum! String theory allows these interactions to occur, not at points, but on tiny strings, giving a value of $\Delta x\neq 0$. According to string theory, the uncertainty principle becomes a function dependent on the string tension:

$$\Delta x=\frac{\hbar}{\Delta p}+\alpha'\frac{\Delta p}{\hbar}\quad\text{where}\quad\alpha'=\frac{1}{2\pi T_0}.$$

The simplest solution to this problem of gauge choice is to choose the light-cone gauge. While the motivation for this choice of gauge is not apparent from the outset, its usefulness is readily realized upon implementation.

If we declare *n* to be a unit vector with no spacelike component, the gauge conditions for τ and σ can be expressed, respectively, as

$$n\cdot X(\tau,\sigma)=\beta\alpha'(n\cdot p)\tau,\quad\text{and}$$

$$(n\cdot p)\sigma=\frac{2\pi}{\beta}\int_0^\sigma d\tilde{\sigma}\,n\cdot P^\tau(\tau,\tilde{\sigma}),$$

where β = 1 or 2, depending on whether we are dealing with an open or closed string.

The beauty of this choice of gauge is that it simplifies the minimization of the variance in the Nambu-Goto string action, yielding the following constraint equations:

$$\dot{X}\cdot X'=0,\quad \dot{X}^2\cdot X'^2=0,$$

which reveal the equations of motion to be given by

$$\boxed{\ddot{X}^\mu-(X^\mu)''=0.}$$

This is easily recognized as the wave equation!

The next task is to find the explicit solution to this equation. We can simplify and generalize our approach into two steps. First, we recognize that the general solution to the wave equation is a linear combination of left and right moving waves, i.e.,

$$X^\mu(\tau,\sigma)=\frac{1}{2}\left(f^\mu(\tau+\sigma)+g^\mu(\tau-\sigma)\right).$$

Next, we can use a Fourier expansion and apply the boundary conditions that arise due to D-branes and parameterization choices. The result of these operations yields a daunting solution for $X(\tau,\sigma)$. However, the messy details of this equation are not as important as the meanings behind its components.

This term tells us where the string is going.

$$X^\mu(\tau,\sigma)=x_0^\mu+\sqrt{2\alpha'}\alpha_0^\mu\tau-i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma.$$

This term tells us how the string is oscillating.

If we imagine a string moving through space-time it will have a certain center-of-mass momentum. It will also possess an energy associated with oscillation. These are exactly the components of our equation!

From Strings to Particles

It is precisely the oscillations of the strings that determine the manifestation of specific particles. The values of the *α's* correspond to the different modes of oscillation, which in turn correspond to the amount of energy that is associated with the wave. Because energy and mass are interchangeable, we get an equivalent mass from these modes. More specifically, the relationship is given by

$$M^2=\frac{1}{\alpha'}\sum_{n=1}^{\infty}n\,a_n^{I*}a_n^I.$$

----- The weakness of gravity -----

One of the big questions in theoretical physics that any unified theory will need to address is the relative weakness of gravity. Gravity is, in fact, weaker than the electromagnetic force by a factor of ~ 10^{30}! According to string theory, this difference is a consequence of the idea that, while the electromagnetic force is confined to our familiar 4-dimensional universe, gravity lives in a much larger space. This larger space accounts for the dilution of gravity and addresses the problem of observing a graviton.

Open strings vs. Closed strings

Open strings have their endpoints connected to D-branes and are, therefore, confined to that dimensionality. An open string can manifest itself as any particle with integer spin. It is for this reason that bosonic string theory is incapable of describing the physical world… there is no accounting for the fermions that make up all of matter!

Having no endpoints, closed strings are not bound to reside in any dimensionality. Closed strings manifest themselves as gravitons, the theorized fundamental force carriers for the gravitational force.

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